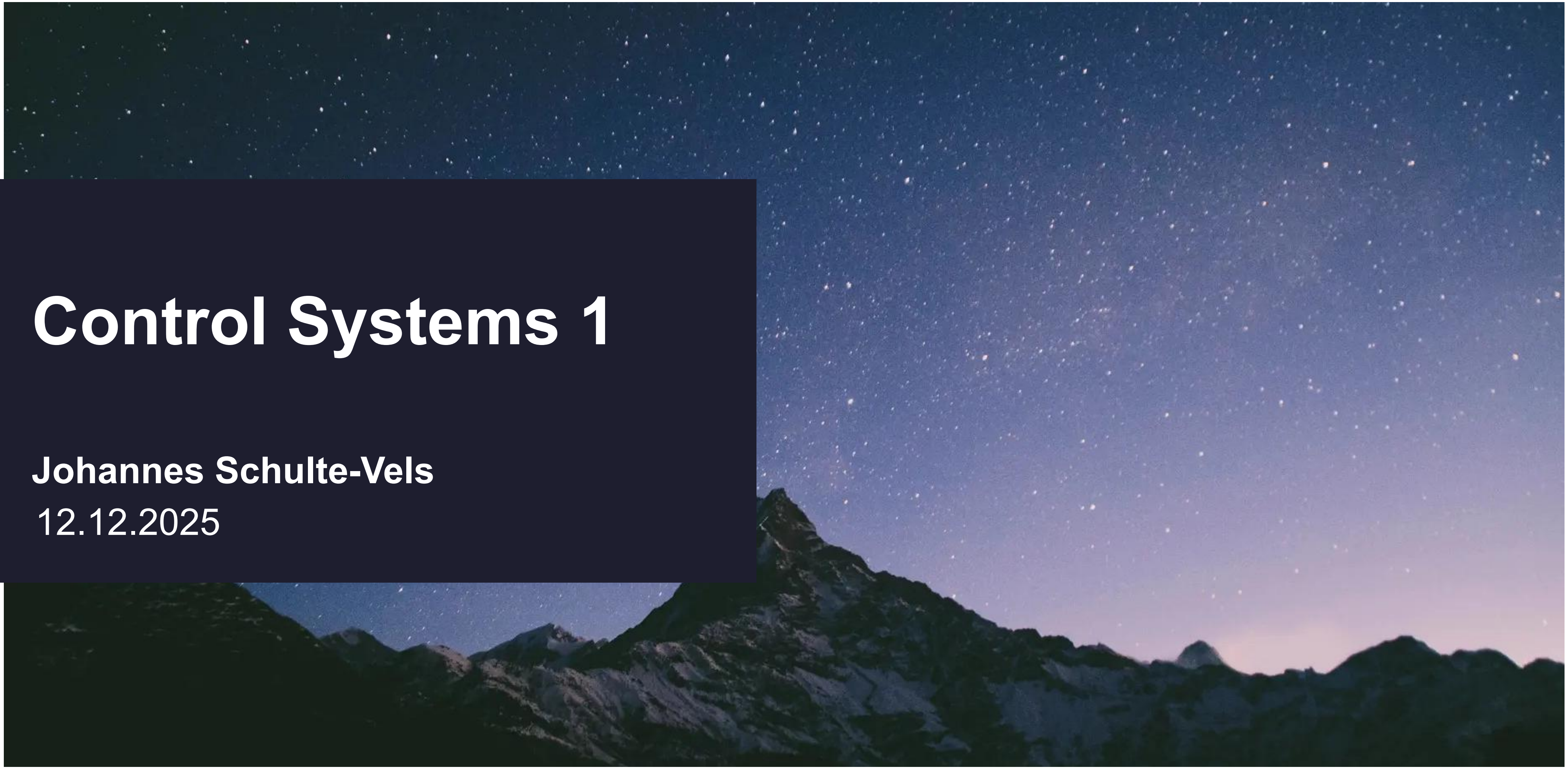
ETH zürich

Institute for Dynamic Systems and Control



Control Systems 1

Johannes Schulte-Vels

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Welcome!

Polybox

PW: jschul

Website

jschultev.github.io/personal_website

Today

- Repetition Session 12
- Theory Recap
  - Nonlinear Systems
  - Absolute Stability
  - Describing Functions and Limit Cycles
- Q&A Session / Done

Repetition Session 12

Time Delay

Real systems have a finite computation time and / or transmission time, meaning the control input has some delay compared to the reference signal.

Time Delays will shift the input signal $u(t)$ by $T \geq 0$ units of time, resulting in the output:

$$\boxed{y(t) = u(t - T)}$$

The Time-Delay operator is **Linear** and **Time-Invariant** so **(LTI)**
To derive the Time-Delay operator and TF we will as usual plug in $u = e^{st}$, which will lead us to:

$$\boxed{y(t) = e^{s(t-T)} = e^{st}e^{-sT} = e^{-sT}u(t)}$$

The TF is therefore given by $\boldsymbol{e^{-sT}}$

Time Delay Operator

The operator e^{-sT} may be LTI, but unfortunately it is **not rational**.

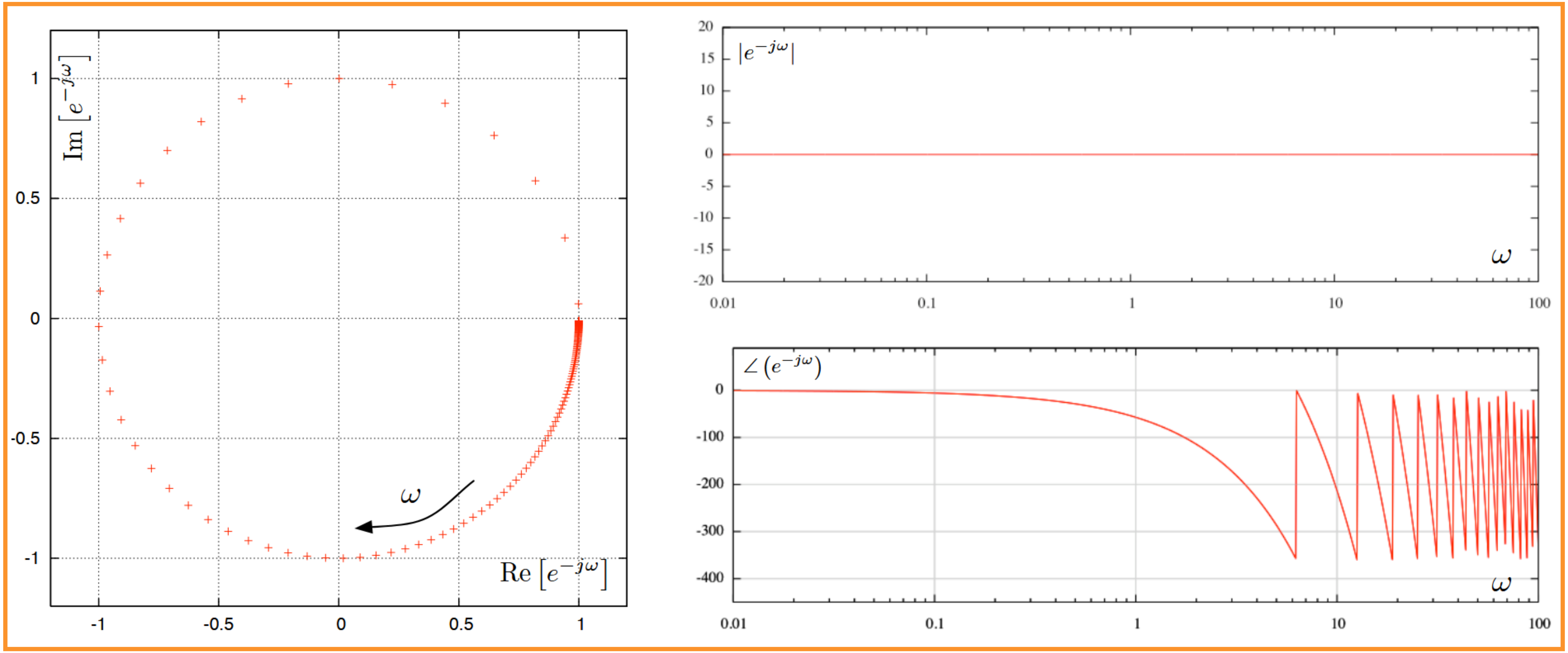
Let's look at the frequency response. Plugging in $s = j\omega$ we get:

$$\left|e^{-j\omega T}\right| = 1 \qquad \angle\left(e^{-j\omega T}\right) = -\omega T$$

From this we can deduce 2 important consequences:

- The **Magnitude is constant** and equal to one for all frequencies
- The **phase decreases linearly** with ω, meaning:
  a sinusoidal signal gets more and more shifted

Time Delay Plot



Time Delay Consequences

$$L'(j\omega) = e^{-j\omega T} L(j\omega)$$

Crossover frequency

We will now ompare L' with L.

Looking at the phase margin, for the system without time delay we find $\varphi_{m,0} = 180^\circ + \angle L(j\omega_c)$

For the time delay system:

$$\varphi_{m,T} = \varphi_{m,0} - \omega_c T$$

- For a fixed delay T, **increasing the crossover frequency ω_c reduces the phase margin.**
- For a fixed crossover frequency, **increasing the delay T reduces the phase margin**.

Padé Approximation

As already mentioned, e^{-sT} is not a rational function. But we would still like to do root locus! Therefore, we will approximate this exponential time delay as a ratio of two polynomials (a.k.a. Padé approximation)

$$e^{-sT} \approx k\,\frac{s+p}{s+q}$$

To get the coefficients we will compare it to the taylor series expansion: $k\frac{s+p}{s+q} = 1 - sT + \frac{1}{2}(sT)^2$

Finally we find:

$$e^{-sT} \approx \frac{\frac{2}{T} - s}{\frac{2}{T} + s}$$

Which means that we can now handle the our TF just as at the beginning, using tools like root locus.

Theory Recap

Nonlinear Systems

Nonlinear Systems

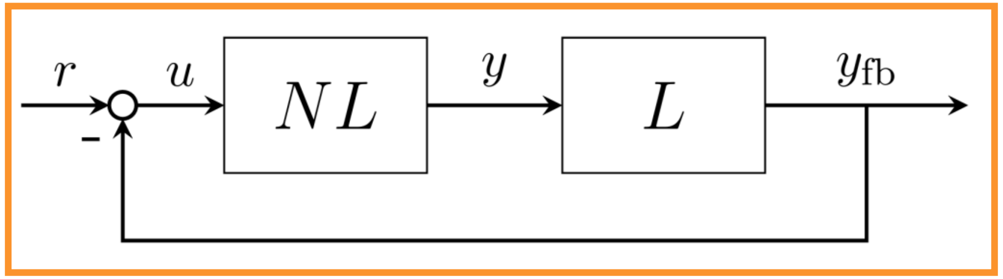
Remember the recap we made last week about nonlinear systems. The common approach was to just **linearize** about a certain point.

This linear models however cannot capture all the phenomena nonlinear ones may exhibit. (For example **limit cycles**, which we will explore today)

So today we will try to briefly go into the analysis of these nonlinear systems!

Nonlinear Systems

Typical nonlinear systems we will consider look like the one below.
$L(s)$ is still the linear open-loop TF, but NL now being the **NonLinearity**. The y_{fb} is then the actual feedback signal.



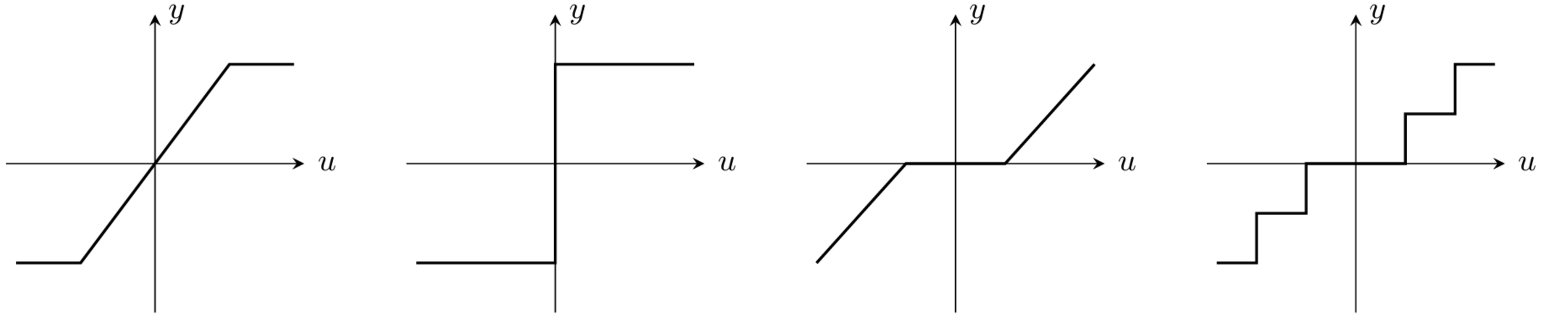
NonLinearities

But what even are those **NonLinearities (NL)**?

They often arise from real physical systems.

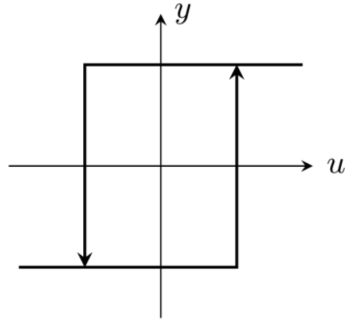
Example: Imagine your Innoprojekt motors for the weels of the robot.
They cannot go infinitely fast. They have a certain torque limit they can provide and anything over that value just cannot get realised. This behaviour would correspond to plot a), the **saturation.**

Static, memoryless ← **What we will look at**



(a) Saturation (b) Switch or Relay (c) Dead Zone (d) Quantizer

Dynamic, with memory



(b) Schmitt Trigger

Absolute Stability

Absolute Stability

Just as for linear systems, we would also like to assess the stability of nonlinear systems.

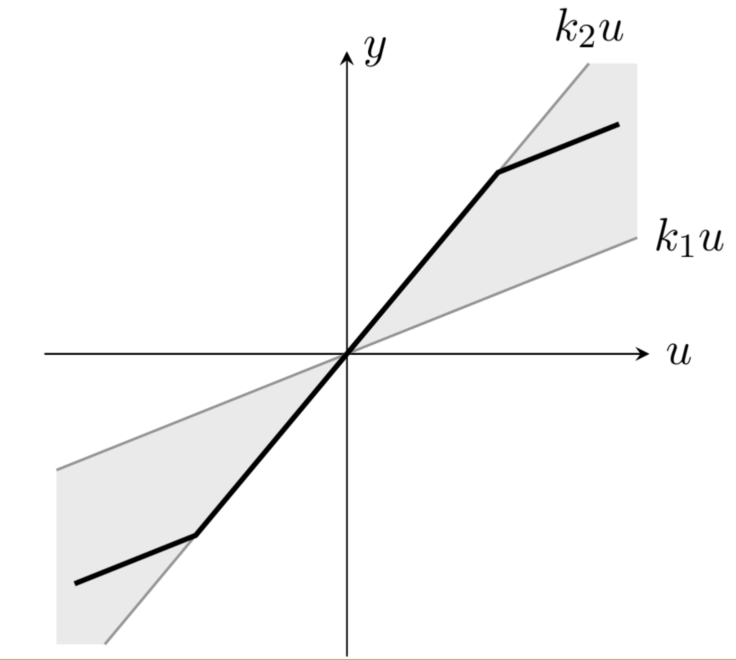
A common and good approach, defining a boundaries that include all values of NL.

$$NL(0) = 0 \quad uk_1 \leq NL(u) \leq uk_2 \quad \forall u \neq 0$$

The closed-loop is said to be absolutely stable, if u = 0 is a globally asymptotically stable equilibrium of the closed-loop system.

Mathematically: for any initial condition of $L(s)$

$$\lim_{t\to\infty} u(t) = 0, \quad \text{and} \quad \lim_{t\to\infty} y(t) = 0$$



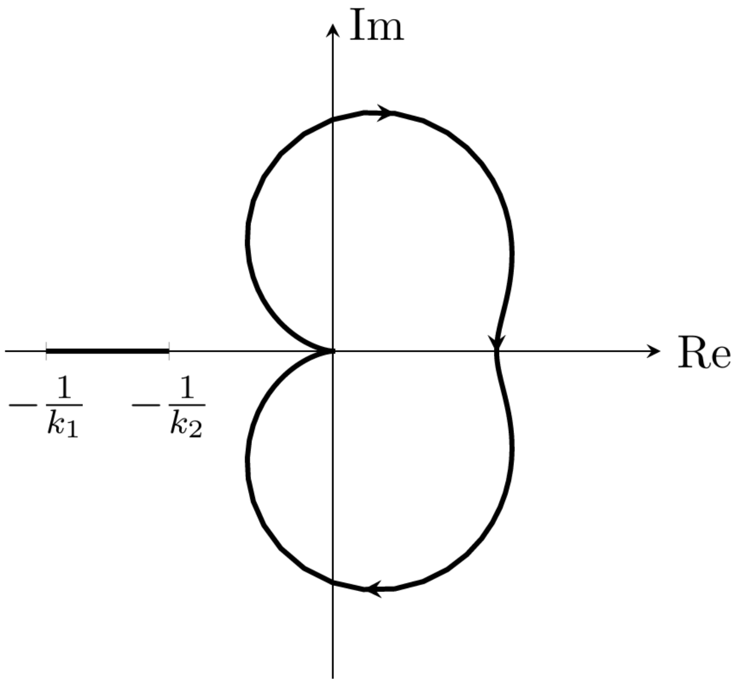
Not super important

Necessary Condition

The way we defined **NL**, it can include all gains $k_1 \leq k \leq k_2$

Now looking at the Niqyuist criterion (at which we look because we are interested in stability), we see that we now not only have to consider $-\frac{1}{k}$, but in the entire interval $[-\frac{1}{k_1}, -\frac{1}{k_2}]$

This means that we actually have to count the correct number of encirclements for the segment $[-\frac{1}{k_1}, -\frac{1}{k_2}]$ in order to **assess stability**

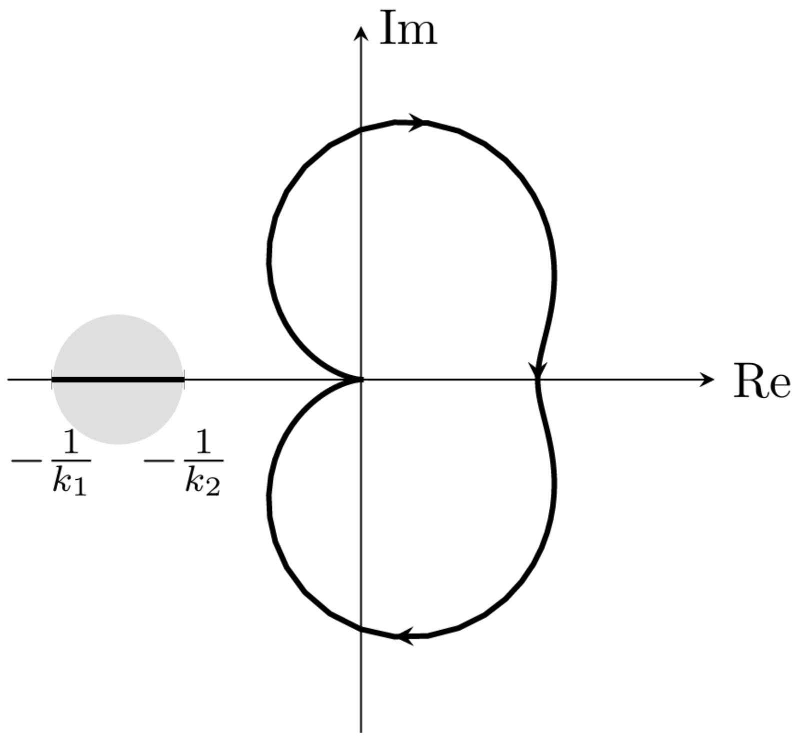


Sufficient Condition, a.k.a Circle Criterion

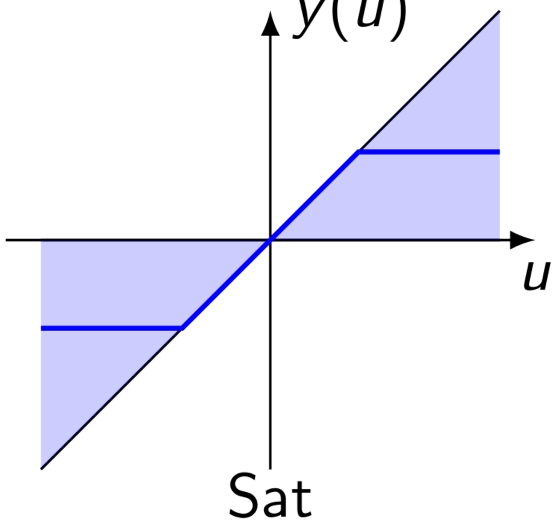
We can be even more careful and define a larger region we have to avoid.
This will lead us to a **circular region** with diameter $[-\frac{1}{k_1}, -\frac{1}{k_2}]$ that we will have to avoid.

This condition is **sufficient**, meaning that if it is fullfilled, we are in fact stable

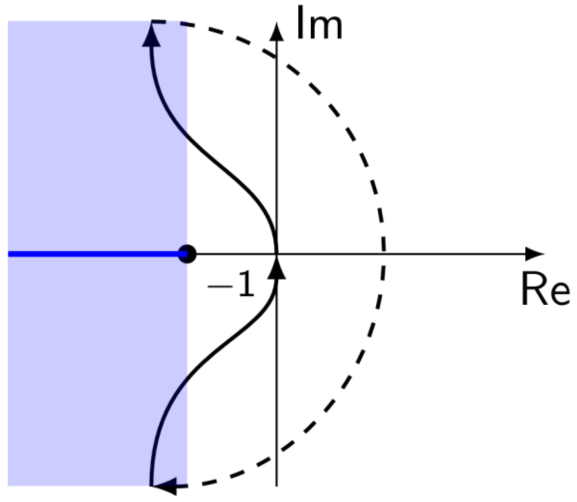
It is however **not necessary** (in contrary to the condition before), meaning that if the condition is not fullfilled, **we cannot make any statement about stability!**



Example



We have a NL with the sector $[\mathbf{0}, \mathbf{1}]$



The segment and circle we may not touch / encircle is therefore $[-\infty, -\mathbf{1}]$

It can be seen, that we touch this region (a circle with infinte diameter, looking like a square).

Since this **only violates the circle criterion** and not the necessary condition, we **cannot make any statement about stability.**

Describing Functions and Limit Cycles

Describing Functions

Let me make some detours. May seem random (which it kinda is), but the important thing is to take certain informations with you.

We saw that we cannot determine stability if the necessary condition is met, but not the sufficient one.

We will introduce **describing functions**, as an **approximate method** to analyze such cases.

Frequency Response

Just as for the linear systems, lets see what happens when looking at the frequency response, meaning we apply a sinusoidal input

$$u(t) = A\sin(\omega t)$$

The output is then some periodic function with the **same frequency as the input**

$$y(t) = f(A\sin(\omega t))$$

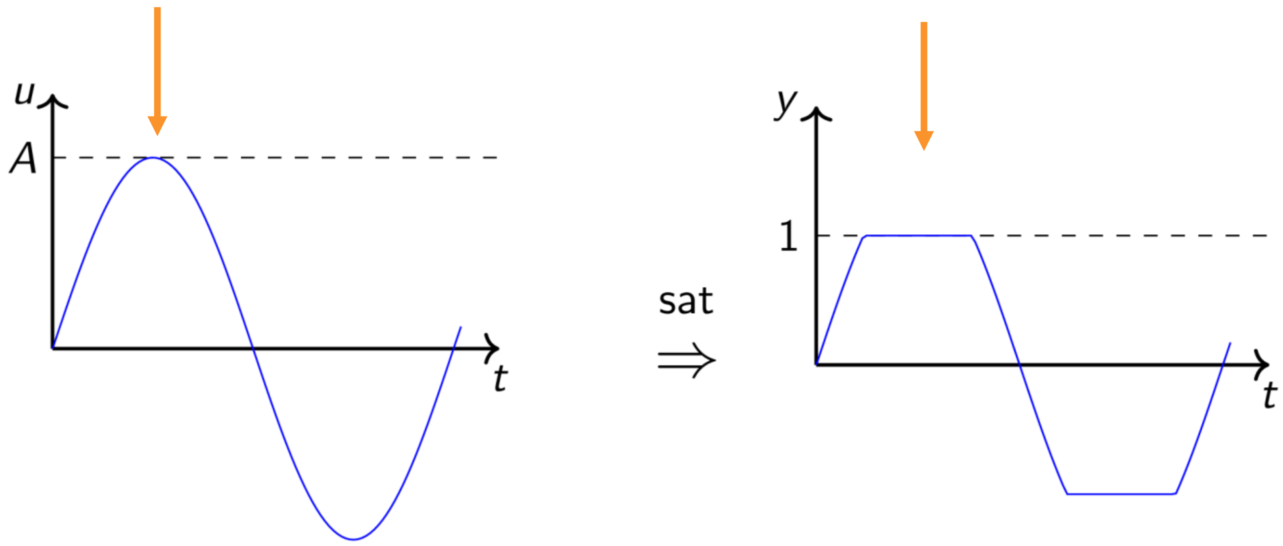
Since the input is periodic, the output will also be periodic. This means that we can decompose the function into ist harmonic components using **Fourier series expansion!**

Example

$$y = \begin{cases} 1 & \text{if } u \geq 1; \\ u & \text{if } -1 < u < 1 \\ -1 & \text{if } u \leq -1. \end{cases}$$

If $A \leq 1$, then $y = u$

If $A > 1$, then $y = 1$



So now lets do **Fourier**

$$y(t) = \frac{a_0}{2} + \sum_{i=1}^{+\infty} [a_i \cos(i\omega t) + b_i \sin(i\omega t)]$$

Coefficients:

$$a_i = \frac{1}{\pi} \int_{-\pi}^{\pi} y(t) \cos(i\omega t) \mathrm{d}(\omega t) \qquad b_i = \frac{1}{\pi} \int_{-\pi}^{\pi} y(t) \sin(i\omega t) \mathrm{d}(\omega t)$$

Attention:
i is the index, not imaginary number j

Example

$$y(t) = \frac{a_0}{2} + \sum_{i=1}^{+\infty} [a_i \cos(i\omega t) + b_i \sin(i\omega t)]$$

Generally for Fourier:

$$\begin{aligned} y(t) \quad \text{odd} \quad &\implies a_i = 0 \quad \forall i \\ y(t) \quad \text{even} \quad &\implies b_i = 0 \quad \forall i \end{aligned}$$

We had an odd system, so therefore we can only consider $\boldsymbol{b_i}$.
Since most real world systems act as **low pass filters**, meaning high frequencies get attenuated (gedämpft),
It is okay to only look at the first order harmonic of the Fourier expansion, meaning we only look at $\boldsymbol{b_1}$

$$y(t) \approx b_1 \sin(\omega t)$$

Now since b_1 is a function of A, we now define the describing function **N(A)** as the ratio $\boldsymbol{\frac{b_1}{A}}$:

$$N(A) = \frac{b_1(A)}{A} = \frac{1}{\pi A} \int_{-\pi}^{\pi} y(t) \sin(i\omega t) \; d(\omega t)$$

Generalizing

Dont worry too much about the content of these slides. Especially the derivation, but also many things arent so important. I just show them for completeness

Generalizing for any input, our old friend e^{st}, with only an imaginary part, so $e^{j\omega t}$, comes into play again: $\boldsymbol{u(t) = Ae^{j\omega t}}$.

The describing function for the general input then becomes

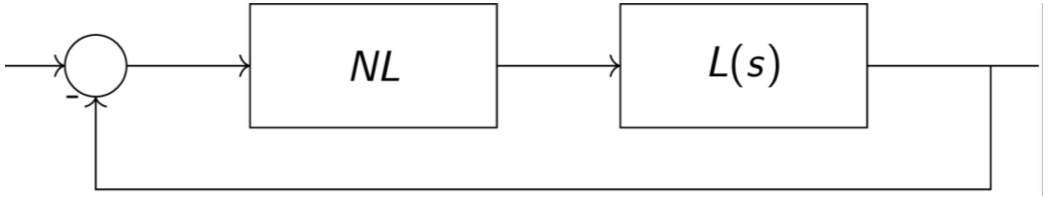
$$N(A,\omega) = \frac{C_1(A,\omega)}{A} e^{j\phi(A,\omega)}$$

$$c_1(A,\omega) = \sqrt{a_1^2 + b_1^2}, \quad \phi(A,\omega) = \arctan\left(\frac{a_1}{b_1}\right)$$

Describing functions are an approximation of the NL as an amplitude-dependent gain!

Limit Cycles

So now we can approximate our NonLinearities with describing functions!

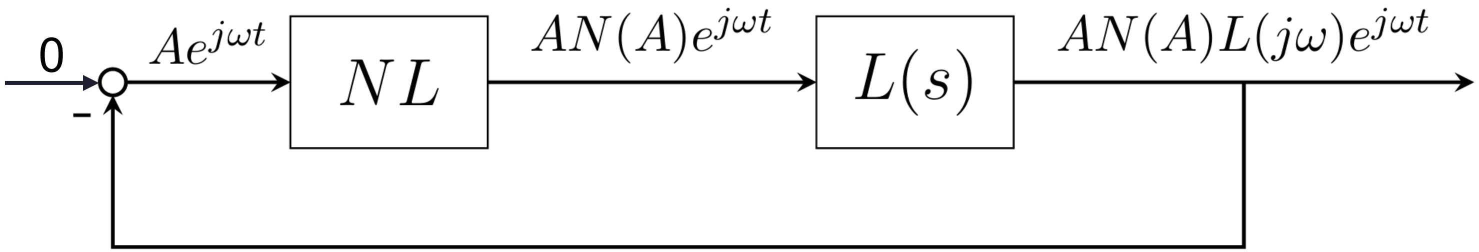


The new open-loop TF can now be written as $\boldsymbol{L^*(s) \approx N(A) \cdot L(s)}$**,** where $N(A)$ is **amplitude dependent.**

What happens now for changing A?

Limit Cycles

When actually applying the general input to the nonlinear system, it looks like below



Since the reference is 0, the error e will just be $0 - y$. So the input will just be $u = -y$.

So what happens if $A = -AN(A)L(j\omega)$? The input will actually always stay the same!
This is called a self sustaining oscillation, and also a **limit cycle!**

A limit cycle fulfills the following equation:

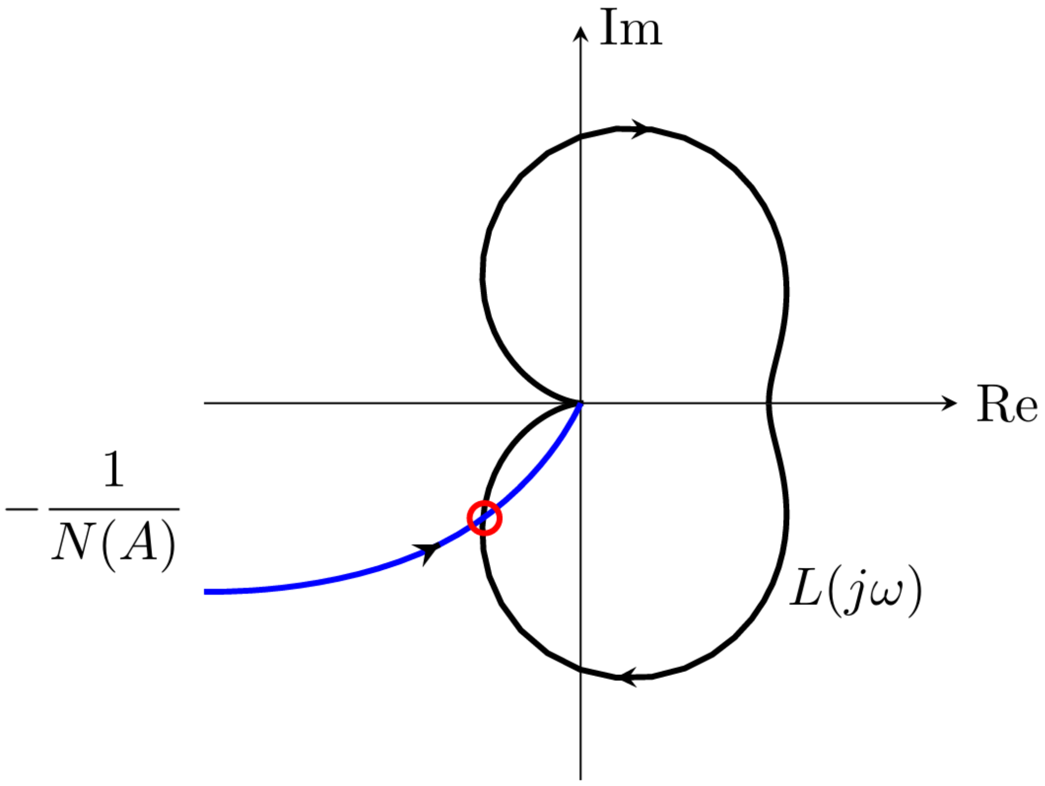
$$-\frac{1}{N(A)} = L(j\omega)$$

Checking for Limit Cycles

When plotting $L(j\omega)$ and $-\frac{1}{N(A)}$ finding intersection points between those plots corresponds to finding limit cycles.

It is nothing but fulfilling the equation from before.

$$-\frac{1}{N(A)} = L(j\omega)$$

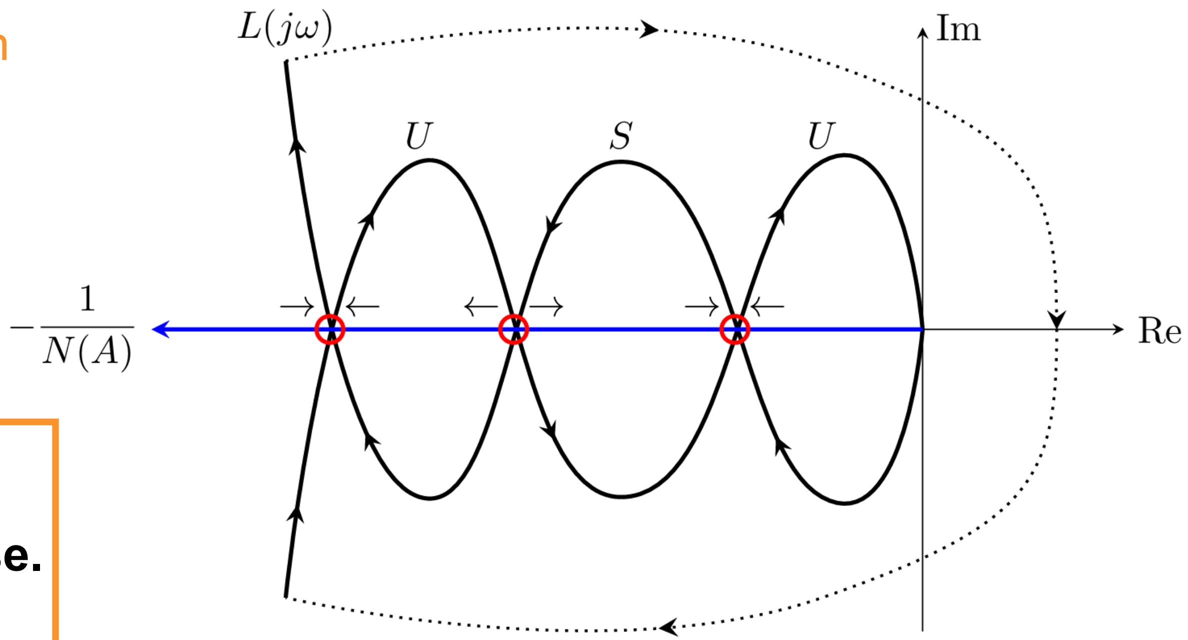


Limit Cycle Stability

Limit Cycles can be
- **Stable**: The system converges to them, and small deviations from the exact amplitude dont matter
- **Unstable**: For small deviations we go away from this limit cycle

To the right is the nyquist plot of the OL-stable $L(j\omega)$, and the plot of $-\frac{1}{N(A)}$. The arrow indicates in which direction the function goes, if A is increased!

You can also see some regions marked.
- **S** stands for **stable**
- **U** stands for **unstable**



If the point $-\frac{1}{N(A)}$ (this is our new $-\frac{1}{k}$) is in:
- **Unstable region:** The amplitude **A** tends to **increase.**
- **Stable region**: The amplitude **A** tends to **decrease**

Limit Cycle Stability

If the point $-\frac{1}{N(A)}$ (this is our new $-\frac{1}{k}$) is in:

- **Unstable region:** The amplitude **A** tends to **increase.**
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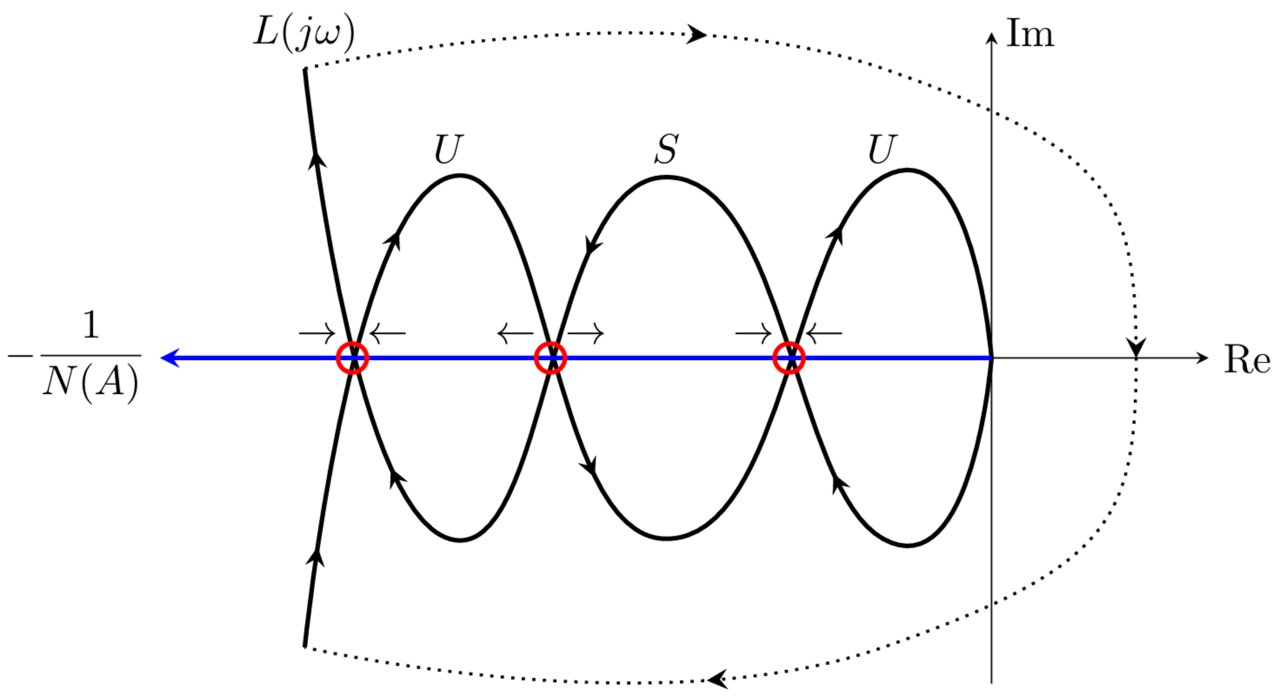

So there are 2 things happening whenyou are located at a limit cycle, and move into a certain direction:

1. You get **pushed back to the limit cycle,** because the region you are in leads to a pushback
   - **Then you are in a stable limit cycle**

2. You **further distance yourself from the limit cycle**, because the region supports your direction.
   - Then you are in an **unstable limit cycle**

What??

These last few steps might feel confusing and too much and weird and whatever…

Dont worry too much, there are (in my opinion, no guarantee, but quite certain) only a couple of things you should take with you from this session:

- What are NonLinearities and why are they there (e.g. physical systems, nonlinear behaviour)
- Necessary and sufficient condition for absolute stability (**if this is not clear pls ask me!**)
  - **Could come in an exam!**
- Know that describing functions describe the NonLinearities with frequency and amplitude
- How to asses stability of limit cycles with given plots (**if this is not clear pls ask me!**)
  - **Could come in an exam!**

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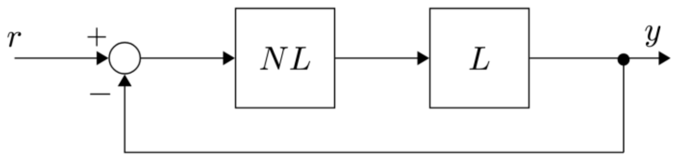
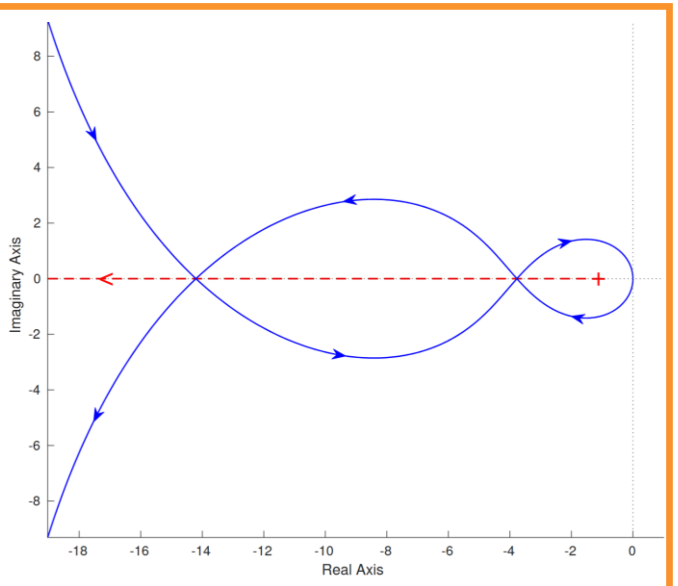


$$N(A) = \frac{2}{\pi}\left[\arcsin\left(\frac{1}{A}\right) + \frac{1}{A}\sqrt{1-\left(\frac{1}{A}\right)^2}\right]$$



L(s) Pole at the origin s = 0 and has one unstable pole

Without doing any stability analysis, what is the **maximum number M** of **potential limit cycles** that the closed loop system T could support?

A) $M = 0$

B) $M \to \infty$

C) $M = 2$

D) $M = 1$

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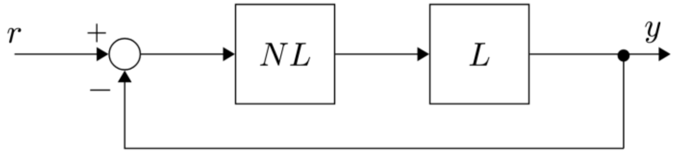

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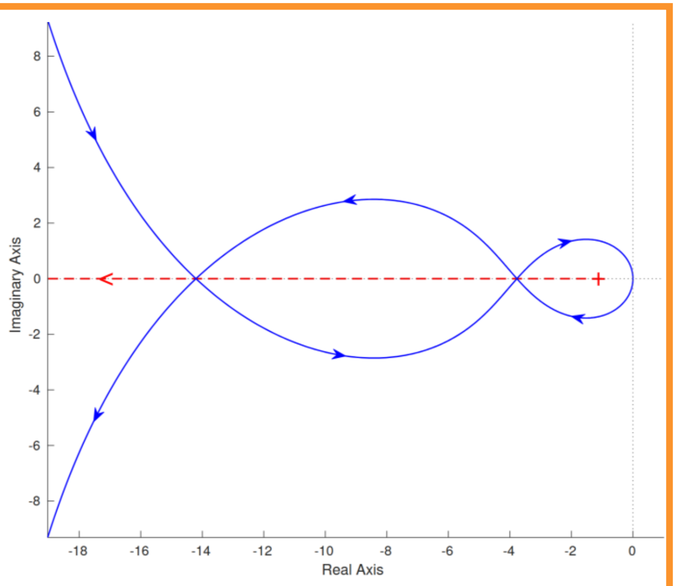

L(s) Pole at the origin s = 0 and has one unstable pole

What is the **number M_s** of **stable limit cycles** that exist in the closed loop system T?

A) $M = 0$

B) $M \to \infty$

C) $M = 2$

D) $M = 1$

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- [A] The circle criterion is a necessary and sufficient condition for absolute stability.
- [B] Describing functions can be seen as approximate transfer functions that represent the transfer function from sinusoidal inputs of amplitude A and frequency ω to the first harmonic of the output of the corresponding non-linearity.
- [C] Limit cycles are typically observed in linear systems.
- [D] The describing function of a static non-linearity is an amplitude dependent gain $N(A)$.

A) B)

C) D)

Sad Goodbye

Wont be around next week!

This was my last session today! ☹
This was however the last important and real exam relevant lecture! So you should be well set!

For a nice Kahoot Quizz next week, you might want to join **Kissan in the ML F40, or Nico in the CAB G59.**

Maybe I will see you again next year as a **Fluid Dynamics Advanced TA.**
It's gonna be interesting…

Thanks to all of you, it was nice having a somewhat constant crowd!!

For the Exam

- I will upload you a small recap of the **most important slides** (in my opinion), hopefully around next Friday, but surely still this year.

- Solve a lot of exams and practice a lot. Your not required to have a deep understanding…

- Try to already **write your own ZF by hand**, especially when solving exams. You might even put entire problems there, considering you have 40 pages…

- Vote for the stuff you want in the given summary of the exam! Some informations that are on there right now are quite useless as you will see!

jschul@ethz.ch

- Feel free to **shoot me a message** anytime you want! I will try to answer as soon as possible. Keep in mind that there will also be an online forum, and there will also be **Exam Question Sessions** during january

Feedback

jschultev.github.io/personal_website/Feedback

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